Ques:- What do you mean by analytic function? Derive Cauchy-Riemann conditions for a function to be analytic. (2015).

or Define analytic function. Obtain necessary and sufficient conditions for a function to be analytic. Examine whether $\ln z$ is analytic? (2017).

or Setup and explain Cauchy- Riemann equations. (2018)

Ans :- Analytic function :- A function $f(z)$ of a complex variable z is said to be analytic at a point z_0 if it is not only differentiable at the point $z=z_0$ but also at every point in some neighbourhood of z_0.

Example: ① $f(z) = |z|^2$ is differentiable at $z=0$ and nowhere else. so the function $f(z)=|z|^2$ is not analytic function at $z=0$.

② $f(z) = z^2$, $f(z) = \sin z$ etc are differentiable at every point so the function $f(z) = z^2$, $f(z)=\sin z$ etc are analytic function everywhere.

* The functions which are analytic every where in the complex plane, are known as entire function.

Necessary and Sufficient conditions for a function to be analytic:

* Necessary conditions:- If the function $f(z) = u(x,y) + iv(x,y)$ is continuous in some neighbourhood of the point z and it is differentiable at the point z then the first order partial derivatives of $u(x,y)$ and $v(x,y)$ exist and satisfy the equations

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \quad \text{——— ①}$$ at the point z.

These equations are known as Cauchy- Riemann equations (conditions) for a function $f(z)$ to be analytic.

Hence if $f(z)$ is analytic in a domain D then the partial derivatives $\frac{\delta u}{\delta x}$, $\frac{\delta v}{\delta x}$, $\frac{\delta u}{\delta y}$ and $\frac{\delta v}{\delta y}$ exist and satisfy the Cauchy-Riemann (C-R) equations

i.e, $$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

Proof :- Let $f(z) = u(x,y) + i\,v(x,y)$ is analytic function at point z.

Since the function $f(z)$ is analytic at point z so the function $f(z)$ will be differentiable at the point z.

i.e, $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists} \quad \text{——— (2)}$$

Since $z = x + iy$ so $\Delta z = \Delta x + i\Delta y$

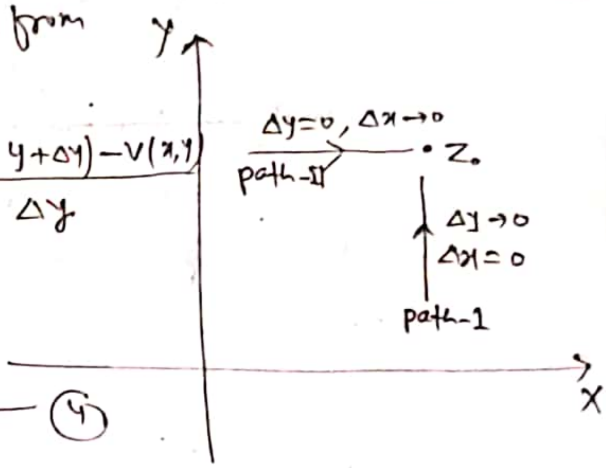
$$f(z) = u(x,y) + i\,v(x,y)$$

$$f(z+\Delta z) = u(x+\Delta x,\ y+\Delta y) + i\,v(x+\Delta x,\ y+\Delta y)$$

From equⁿ (2)

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta x + i\Delta y} \text{ exists} \quad \text{——— (3)}$$

path 1 :- If $\Delta x = 0$ and $\Delta y \to 0$ then from equⁿ (3) (parallel to y axis)



$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$\Rightarrow f'(z) = \frac{1}{i}\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

$$\Rightarrow f'(z) = -i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \quad \text{——— (4)}$$

path II :- If $\Delta y = 0$ and $\Delta x \to 0$ (parallel to X axis) then from equⁿ (3)

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\cdot\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \quad \text{——— (5)}$$

From equⁿˢ (4) and (5)

$$-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$$

Equating real and imaginary parts from bothsides, we get

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \quad \text{proved.}$$

C-R equations are necessary conditions for a function $f(z)$ to be analytic but C-R equations are not sufficient condition for a function to be analytic or differentiable.

Sufficient condition for a function $f(z)$ to be analytic.

If $u(x,y)$, $v(x,y)$ and their partial derivatives with respect to x and y are continuous and satisfy C-R equations in a domain D then the function $f(z)$ will be analytic at all points inside the domain D (not necessarily on the boundaries). Thus for existence of derivatives of function $f(z)$, the sufficient requirement is continuity of these derivatives satisfying C-R equations $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$.

Proof:— Let $W = f(z) = u(x,y) + iv(x,y)$ —— (6)

Now $\Delta W = \Delta f(z) = \Delta u + i\Delta v$

$$\Rightarrow \Delta f(z) = \frac{\Delta u}{\Delta x}\cdot\Delta x + \frac{\Delta u}{\Delta y}\cdot\Delta y + i\frac{\Delta v}{\Delta x}\cdot\Delta x + i\frac{\Delta v}{\Delta y}\cdot\Delta y$$

$$\Rightarrow \Delta f(z) = \left(\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}\right)\cdot\Delta x + \left(\frac{\Delta u}{\Delta y} + i\frac{\Delta v}{\Delta y}\right)\Delta y \quad \text{—— (7)}$$

The derivative of $f(z)$ with respect to z is given by

$$\frac{df(z)}{dz} = f'(z) = \lim_{\Delta z\to 0}\frac{\Delta f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\left(\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}\right)\Delta x + \left(\frac{\Delta u}{\Delta y} + i\frac{\Delta v}{\Delta y}\right)\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\left(\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}\right) + i\frac{\Delta y}{\Delta x}\left(\frac{\Delta v}{\Delta y} + \frac{1}{i}\frac{\Delta u}{\Delta y}\right)}{1 + i\frac{\Delta y}{\Delta x}}$$

$$\frac{df(z)}{dz} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}}\left[1 + i\frac{\Delta y}{\Delta x}\cdot\frac{\frac{\Delta v}{\Delta y} - i\frac{\Delta u}{\Delta y}}{\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}}\right] \quad \text{—— (8)}$$

From C-R equations, $\frac{\Delta u}{\Delta x} = \frac{\Delta v}{\Delta y}$ and $\frac{\Delta u}{\Delta y} = -\frac{\Delta v}{\Delta x}$

put in eqn (8)

$$\frac{df(z)}{dz} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}} \cdot \left[1 + i\frac{\Delta y}{\Delta x} \cdot \frac{\frac{\Delta v}{\Delta y} + i\frac{\Delta y}{\Delta x}}{\frac{\Delta v}{\Delta y} + i\frac{\Delta v}{\Delta x}} \right]$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}}{\left(1 + i\frac{\Delta y}{\Delta x}\right)} \cdot \left[1 + i\frac{\Delta y}{\Delta x}\right]$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}$$

$$\Rightarrow \frac{df(z)}{dz} = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = \frac{\delta f}{\delta x} \quad \text{——— (9)}$$

eqn (9) shows that derivatives of $f(z)$ with respect to z is independent of the direction of approach in the complex plane as long as derivatives are continuous. If derivatives are not continuous then $\frac{\delta u}{\delta x}$, $\frac{\delta u}{\delta y}$, $\frac{\delta v}{\delta x}$ and $\frac{\delta v}{\delta y}$ would have different values depending upon how derivatives were evaluated and function would not be analytic. eqn (9) states that if these partial derivatives are continuous then $f'(z)$ exists and possesses a finite unique value. i.e, the function $f(z)$ is necessarily analytic.